EXPONENTIAL SUMS INVOLVING THE MÖBIUS FUNCTION

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ABSTRACT. Let $\mu(n)$ be the möbius function, $e(x) = e^{2\pi i x}$, x real. This paper gives the estimate of exponential sums involving the möbius function

$$S_k(x,\alpha) = \sum_{n\le x} \mu(n) e(n^k \alpha)$$

under the weak Generalized Riemann Hypothesis when $k \ge 3$.

1. Introduction and statement of results

Let $\mu(n)$ be the möbius function, $e(x) = e^{2\pi i x}$, $k \ge 1$ an integer, x real. The estimate of the exponential sum

$$S_k(x,\alpha) = \sum_{n\le x} \mu(n) e\left(n^k \alpha\right) \tag{1.1}$$

was first studied by Davenport [2] in 1937 with Vinogradov's elementary method. He proved that for any $A > 0$

$$\max_{\alpha\in[0,1]} |S_1(x,\alpha)| \ll_A x(\log x)^{-A}. \tag{1.2}$$

Here and in the sequel $\ll_A$ indicates that the implied constant depends at most on A. For $k \ge 2$, Hua [4] proved that

$$\max_{\alpha\in[0,1]} |S_k(x,\alpha)| \ll_A x(\log x)^{-A}$$

holds for any $A > 0$.

Now we consider that the estimate of exponential sums under the following weak Generalized Riemann Hypothesis(briefly GRH), for some $0 \le \delta < \frac{1}{2}$ and every Dirichlet character χ,

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \text{ has no zeros in the half plane } \sigma > \frac{1}{2} + \delta,\ (s = \sigma + it). \tag{1.3}$$

For $k = 1$, The best result in this direction is due to Baker and Harman [1], who showed in 1991 that for any $\varepsilon > 0$

$$\max_{\alpha\in[0,1]} |S_1(x,\alpha)| \ll_\varepsilon x^{b+\varepsilon}, \tag{1.4}$$

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where

$$b = \begin{cases} \delta + \frac{3}{4}, & \text{for } 0 \le \delta < \frac{1}{20}; \\ \frac{4}{5}, & \text{for } \frac{1}{20} \le \delta < \frac{1}{10}; \\ \frac{1}{2}\delta + \frac{3}{4}, & \text{for } \frac{1}{10} \le \delta < \frac{1}{2}. \end{cases}$$

For the case $k \ge 2$, Liu and Zhan [6] proved that for any $k \ge 2$, $\varepsilon > 0$, under GRH,we have

$$\max_{\alpha \in [0,1]} |S_k(x,\alpha)| \ll_\varepsilon x^{\varphi_k + \varepsilon},$$

where

$$\varphi_k = 1 - \frac{1}{2^{2k-1}}.$$

In this paper we combine Ren [7], Kumchev [5], Wooley [10] and Zhao [11] to improve the result of Liu and Zhan [6]. When q is small, we use analytic method to get our result. When q is large, Kumchev, Wooley and Zhao's results are much better. Our main result of this paper is the following theorem.

Theorem 1. *For any $k \ge 3$, and $\varepsilon > 0$, under weak GRH, then we have*

$$\max_{\alpha \in [0,1]} |S_k(x,\alpha)| \ll_\varepsilon x^{b_k + \varepsilon},$$

where

$$b_k = \begin{cases} 1 - \rho_k + \varepsilon, & \text{if } 0 \le \delta < \frac{1}{2} - k\rho_k, \\ 1 - \frac{1}{2k}(1 - 2\delta) + \varepsilon, & \text{if } \frac{1}{2} - k\rho_k \le \delta < \frac{1}{2} - 3\rho_k, \\ 1 - \frac{1-2\delta}{2^{2k-1}} + \varepsilon, & \text{if } \frac{1}{2} - 3\rho_k \le \delta < \frac{1}{2}, \end{cases} \tag{1.5}$$

and

$$\rho_k = \begin{cases} \frac{1}{3 \times 2^{k-1}}, & \text{if } 3 \le k \le 7, \\ \frac{1}{6k(k-2)}, & \text{if } k \ge 8, \end{cases} \tag{1.6}$$

Remark 1. When $0 \le \delta < \frac{1}{2} - k\rho_k$, the upper bound of $S_k(x,\alpha)$ is independent with δ. In particular, when $\delta = 0$, we get

$$\max_{\alpha \in [0,1]} |S_k(x,\alpha)| \ll_\varepsilon x^{\phi_k + \varepsilon},$$

where

$$\phi_k = \begin{cases} 1 - \frac{1}{3 \times 2^{k-1}}, & \text{if } 3 \le k \le 7, \\ 1 - \frac{1}{6k(k-2)}, & \text{if } k \ge 8, \end{cases}$$

under GRH which improves the result of Liu and Zhan [6].

Notation. Throughout the paper, the letter ε denotes a sufficiently small positive real number, it may be different at each occurrence. For example, we may write

$$x^\varepsilon x^\varepsilon \ll x^\varepsilon.$$

Any statement in which ε occurs holds for each positive ε, and any implied constant in such a statement is allowed to depend on ε. The letter p, with or without subscripts, is reserved for prime numbers. We write $(a,b)=\gcd(a,b)$, and we use $m\sim M$ as an abbreviation for the condition $M<m\le 2M$.

2. Outline of the proof

Take

$$P_1=x^{1/2-\delta},\qquad P_2=x^{1/2+\delta},\qquad Q=x^{k+\delta-1/2}.$$

For $\alpha\in[0,1]$, by Dirichlet's lemma on rational approximations, we can write

$$\alpha=\frac{a}{q}+\lambda,\ \text{with}\ (a,q)=1,\ 1\le a\le q,\ 1\le q\le Q,\ |\lambda|\le\frac{1}{qQ}. \tag{2.1}$$

So for all $\alpha\in[0,1]$ can be divided into three disjoint sets

$$E_1=\left\{\alpha;\alpha=\frac{a}{q}+\lambda,(a,q)=1,\ 1\le q\le P_1,\ |\lambda|\le\frac{1}{qQ}\right\},$$
$$E_2=\left\{\alpha;\alpha=\frac{a}{q}+\lambda,(a,q)=1,\ P_1<q\le P_2,\ |\lambda|\le\frac{1}{qQ}\right\},$$
$$E_3=\left\{\alpha;\alpha=\frac{a}{q}+\lambda,(a,q)=1,\ P_2<q\le Q,\ |\lambda|\le\frac{1}{qQ}\right\},$$

For $\alpha\in E_1,\alpha\in E_2,\alpha\in E_3$, we have the following three propositions, by which we can prove Theorem 1.

Proposition 1. *Assume weak GRH and $k\ge 3$. Then we have*

$$\max_{\alpha\in E_1}|S_k(x,\alpha)|\ll x^{1-\frac{1}{2k}(1-2\delta)+\varepsilon}. \tag{2.2}$$

Proposition 2. *Assume weak GRH and $k\ge 3$. Then we have*

$$\max_{\alpha\in E_2}|S_k(x,\alpha)|\ll x^{c+\varepsilon}, \tag{2.3}$$

where

$$c=\begin{cases}\frac{4}{5}, & \text{if } 0\le\delta<\frac{1}{10},\\ \frac{3}{4}+\frac{\delta}{2}, & \text{if } \frac{1}{10}\le\delta<\frac{1}{2}.\end{cases}$$

Remark 2. When $\alpha\in E_2$, the upper bound of $S_k(x,\alpha)$ is independent with k.

Proposition 3. *Assume weak GRH and $k\ge 3$. Then we have*

$$\max_{\alpha\in E_3}|S_k(x,\alpha)|\ll x^{d_k+\varepsilon}, \tag{2.4}$$

where

$$d_k=\begin{cases}1-\rho_k, & \text{if } 0\le\delta<\frac{1}{2}-3\rho_k,\\ 1-\frac{1-2\delta}{2^{2k-1}}, & \text{if } \frac{1}{2}-3\rho_k\le\delta<\frac{1}{2},\end{cases}$$

and ρ_k is defined in (1.6).

Proof of Theorem 1. From Propositions 1, 2 and 3, we can easily get Theorem 1. □

3. Proof of Proposition 1

We use analytic method to prove Proposition 1. To do this, we need the following lemmas.

Lemma 1. *Let $k \geq 3$, $\alpha = \frac{a}{q} + \lambda$, $(a, q) = 1$. Then for any $\varepsilon > 0$*

$$S_k(x, \alpha) \ll q^{\eta_k + \varepsilon} \sum_{d|q} \max_{\chi_{q/d}} \Bigg| \sum_{\substack{m \leq x/d \\ (m,q)=1}} \mu(m)\chi(m)e(m^k d^k \lambda) \Bigg|,$$

where $\eta_k = 1 - \frac{1}{k}$.

Proof. See [6, Lemma 2]. □

Lemma 2. *Under weak GRH, we have*

$$L^{-1}(\sigma + it, \chi) \ll q^{\varepsilon}(|t| + 1)^{\varepsilon},$$

for $\sigma \geq \frac{1}{2} + \delta + \varepsilon$ and every Dirichlet character $\chi(\text{mod } q)$.

Proof. See [9, Theorem 14.2]. □

Lemma 3. *Assume weak GRH, $k \geq 3$ and $\alpha \in E_1$. Then we have*

$$S_k(x, \alpha) \ll q^{\eta_k} x^{1/2+\delta+\varepsilon}(1 + |\lambda|^{1/2} x^{k/2}), \tag{3.1}$$

where η_k is defined in Lemma 1.

Proof. By Lemma 1 we know that Lemma 3 will follow if we can prove that for any $\varepsilon > 0$ and $d|q$

$$\sum_{\substack{m \sim x/d \\ (m,q)=1}} \mu(m)\chi(m)e(m^k d^k \lambda) \ll d^{-1/2} x^{1/2+\delta+\varepsilon}(1 + |\lambda|^{1/2} x^{k/2}) \tag{3.2}$$

holds uniformly for all $\chi = \chi_{q/d}$.

Let I_1 denote the left-hand side of (3.2), and

$$F(s, \chi) = F_q(s, \chi) = \sum_{\substack{m=1 \\ (m,q)=1}}^{\infty} \mu(m)\chi(m)m^{-s}, \ \sigma > 1$$

$$H(s, \chi) = H_q(s, \chi) = \prod_{p|q} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Then

$$F(s, \chi) = L^{-1}(s, \chi)H(s, \chi). \tag{3.3}$$

By (3.3) we know that under weak GRH the function $F(s, \chi)$ is analytic in the region $\text{Re}(s) \geq \frac{1}{2} + \delta + \varepsilon$ for any $\varepsilon > 0$. Furthermore,

$$H(s, \chi) \ll \prod_{p|q} \left(1 - \frac{1}{\sqrt{p}}\right)^{-1} \ll q^{\varepsilon}, \quad \text{Re}(s) \geq \frac{1}{2} + \delta + \varepsilon. \tag{3.4}$$

By Perron's summation formula we have for $u \le x$

$$\sum_{\substack{m\le u\\(m,q)=1}} \mu(m)\chi(m) = \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} F(s,\chi)\frac{u^s}{s}ds + O\bigg(\frac{x^{1+\varepsilon}}{T} + \log x\bigg).$$

Take $T = x^k$ and shift the path of integration above to $\mathrm{Re}(s) = \frac{1}{2}+\delta+\varepsilon$.

$$\sum_{\substack{m\le u\\(m,q)=1}} \mu(m)\chi(m) = \frac{1}{2\pi}\int_{-x^k}^{x^k} F\bigg(\frac{1}{2}+\delta+\varepsilon+it,\chi\bigg)\frac{u^{\frac{1}{2}+\delta+\varepsilon}}{\frac{1}{2}+\delta+\varepsilon}dt + O(x^{\varepsilon}).$$

Then

$$\begin{aligned}
I_1 &= \int_{x/(2d)}^{x/d} e(d^k u^k\lambda) d\bigg(\sum_{\substack{m\le u\\(m,q)=1}} \mu(m)\chi(m)\bigg)\\
&= \frac{1}{2\pi}\int_{-x^k}^{x^k} F\bigg(\frac{1}{2}+\delta+\varepsilon+it,\chi\bigg)\int_{x/(2d)}^{x/d} u^{-1/2+\delta+\varepsilon/2} e\bigg(d^k u^k\lambda + \frac{t}{2\pi}\log u\bigg)dudt\\
&\quad + O(|\lambda| x^{k+\varepsilon} + x^{\varepsilon})\\
&\ll d^{-1/2-\delta}\int_{-x^k}^{x^k}\bigg|F\bigg(\frac{1}{2}+\delta+\varepsilon+it,\chi\bigg)\bigg|\\
&\quad \times\bigg|\int_{x^k/2^k}^{x^k} v^{-1+1/(2k)+\delta/k+\varepsilon/(2k)} e\bigg(v\lambda + \frac{t}{2k\pi}\log v\bigg)dv\bigg|dt + O(|\lambda| x^{k+\varepsilon} + x^{\varepsilon}).
\end{aligned}$$

Since

$$\bigg(v\lambda + \frac{t}{2k\pi}\log v\bigg)' = \frac{t+2k\pi v}{2k\pi v} \gg \frac{\min_{x^k/2^k\le v\le x^k} |t+2k\pi v|}{x^k}$$

$$-\bigg(v\lambda + \frac{t}{2k\pi}\log v\bigg)'' = \frac{t}{2k\pi v^2} \gg \frac{|t|}{x^{2k}},$$

by Lemma 2 and (3.4), we get

$$\begin{aligned}
I_1 &\ll d^{-1/2-\delta} x^{1/2+\delta+\varepsilon}\int_{-x^k}^{x^k}\bigg|F\bigg(\frac{1}{2}+\delta+\varepsilon+it,\chi\bigg)\bigg|\\
&\quad \times \min\bigg(\frac{1}{\sqrt{|t|+1}}, \frac{1}{\min_{x^k/2^k\le v\le x^k}|t+2k\pi v|}\bigg)dt + O(|\lambda| x^{k+\varepsilon} + x^{\varepsilon})\\
&\ll d^{-1/2-\delta} x^{1/2+\delta+\varepsilon}\int_{-x^k}^{x^k}\min\bigg(\frac{1}{\sqrt{|t|+1}}, \frac{1}{\min_{x^k/2^k\le v\le x^k}|t+2k\pi v|}\bigg)dt\\
&\quad + O(|\lambda| x^{k+\varepsilon} + x^{\varepsilon}).
\end{aligned}$$

On noting that
$$|\lambda|x^k \le d^{-1/2}|\lambda|^{1/2}x^{(k+1)/2},$$
it suffices now to show that
$$\int_{-x^k}^{x^k} \min\left(\frac{1}{\sqrt{|t|+1}}, \frac{1}{\min\limits_{x^k/2^k\le v\le x^k} |t+2k\pi v|}\right) dt \ll (1+|\lambda|^{1/2}x^{k/2})\log x. \tag{3.5}$$
Denote by I_2 the left-hand side of (3.5). If $|\lambda| > x^{-k}$, then
$$\begin{aligned} I_2 &\ll \int\limits_{|t|\le 2^{-k}\pi|\lambda|x^k} \frac{dt}{|\lambda|x^k} + \int\limits_{4k\pi|\lambda|x^k<|t|\le x^k} \frac{dt}{|t|} + \int\limits_{2^{-k}\pi|\lambda|x^k<|t|\le 4k\pi|\lambda|x^k} \frac{dt}{\sqrt{|t|+1}} \\ &\ll \log x + |\lambda|^{1/2}x^{k/2}. \end{aligned}$$
If $|\lambda| \le x^{-k}$, we have that
$$I_2 \ll \int\limits_{|t|\le 4k\pi} 1dt + \int\limits_{4k\pi<|t|\le x^k} \frac{dt}{|t|} \ll \log x.$$
This proves (3.5), and the result follows. □

Proof of Proposition 1. Applying Lemma 3 on E_1, we prove Proposition 1. □

4. Proof of Proposition 2

Ren [7] use analytic method to get a new type upper bound of exponential sums.

Lemma 4 (Ren)**.** *Fix $k \ge 1$, and let $\beta_k = 1/2 + \log k/\log 2$. We have*
$$S_k(x,\alpha) \ll (d(q))^{\beta_k}(\log x)^c\left(x^{1/2}\sqrt{q(1+|\lambda|x^k)} + x^{4/5} + \frac{x}{\sqrt{q(1+|\lambda|x^k)}}\right),$$

Proof. See [7, Theorem 1.1]. □

Remark 3. By [8], we can replace the middle term $x^{4/5}$ by $x^{3/4+\varepsilon}$ under GRH.

Proof of Proposition 2. Applying Lemma 4 on E_2, we prove Proposition 2. □

5. Proof of Proposition 3

We combine Kumchev [5] with Wooley [10], and then we get the following result.

Lemma 5. *Let $k \ge 4$, $\rho = \rho_k$ is defined in (1.6) and suppose that α satisfies (2.1) with $Q = x^{\frac{k^2-2k\rho}{2k-1}}$. Then*
$$S_k(x,\alpha) \ll x^{1-\rho+\varepsilon} + \frac{q^\varepsilon x L^c}{\sqrt{q(1+|\lambda|x^k)}}, \tag{5.1}$$
where the implied constant depends at most on k and ε.

Proof. See [5, Theorem 3] and [10, Theorem 11.1]. □

The next result is due to Zhao [11]. When $k = 3$, he gives a better upper bound and enlarge the value of Q. We also can enlarge the value of Q in Lemma 5 when $k \geq 4$.

Lemma 6. *Suppose that α satisfies (2.1) and $x^{1/2} \leq Q \leq x^{5/2}$, then one has*

$$S_3(x,\alpha) \ll x^{1-1/12+\varepsilon} + \frac{q^{-1/6}x^{1+\varepsilon}}{\sqrt{(1+x^3|\lambda|)}}.$$

Proof. See [11, Lemma 8.5]. □

Remark 4. Following the proof of Lemma 6, we can prove that when $x^{1/2} \leq Q \leq x^{17/6-\varepsilon}$, Lemma 6 is also true. This will be used in our result.

Lemma 7. *Let $k \geq 4$, $\rho = \rho_k$ is defined in (1.6) and suppose that α satisfies (2.1) with $x^{2\rho+\varepsilon} \leq Q \leq x^{k-2\rho-\varepsilon}$. Then*

$$S_k(x,\alpha) \ll x^{1-\rho+\varepsilon} + \frac{q^{\varepsilon}xL^c}{\sqrt{q(1+|\lambda|x^k)}}, \tag{5.2}$$

where the implied constant depends at most on k and ε.

Proof. For any $\alpha \in [0,1]$, there exist $b \in \mathbb{Z}$ and $r \in \mathbb{N}$ with

$$(b,r) = 1,\ 1 \leq r \leq x^{\frac{k^2-2k\rho}{2k-1}} \text{ and } |r\alpha - b| \leq x^{-\frac{k^2-2k\rho}{2k-1}}.$$

So we have

$$S_k(x,\alpha) \ll x^{1-\rho+\varepsilon} + \frac{r^{\varepsilon}xL^c}{\sqrt{r(1+|\alpha-b/r|x^k)}}. \tag{5.3}$$

We assume that

$$r \leq x^{2\rho-\varepsilon} \text{ and } |\alpha - b/r| \leq r^{-1}x^{2\rho-k-\varepsilon}. \tag{5.4}$$

Otherwise, we have $S_k(x,\alpha) \ll x^{1-\rho+\varepsilon}$ by (5.3). Combining (2.1) and (5.4), we have

$$|bq - ar| = |q(b-r\alpha) + r(q\alpha - a)| \leq qr|\frac{b}{r} - \alpha| + qr|\frac{a}{q} - \alpha|$$

$$\leq Qx^{2\rho-k-\varepsilon} + \frac{x^{2\rho-\varepsilon}}{Q} < 1,$$

provided that $x^{2\rho+\varepsilon} \leq Q \leq x^{k-2\rho-\varepsilon}$, hence

$$a = b,\ q = r.$$

We complete the proof. □

When δ is large, we can not use Lemma 6 and Lemma 7 for $\alpha \in E_3$, but we can use the next lemma unconditionally.

Lemma 8. *For $k \geq 3$ and $\alpha \in E_3$ we have unconditionally that*

$$\max_{\alpha\in E_3} |S_k(x,\alpha)| \ll x^{1+\varepsilon}\left(\frac{1}{q} + \frac{1}{x^{1/2}} + \frac{q}{x^k}\right)^{2^{2-2k}}.$$

Proof. See [3, Theorem 1]. □

Proof of Proposition 3. When $0 \le \delta < \frac{1}{2} - 3\rho_k$, for $k = 3$, applying Lemma 6 on E_3, we prove Proposition 3; for $k \ge 4$, applying Lemma 7 on E_3, we prove Proposition 3.

When $\frac{1}{2} - 3\rho_k \le \delta < \frac{1}{2}$, applying Lemma 8 on E_3, we prove Proposition 3. □

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