EXPONENTIAL SUMS OVER PRIMES IN SHORT INTERVALS

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Abstract. Let $\Lambda(n)$ be the von Mangoldt function, x real and $2 \leq y \leq x$. This paper improves the estimate on the exponential sum over primes in short intervals

$$S_k(x,y;\alpha) = \sum_{x<n\leq x+y} \Lambda(n) e\left(n^k\alpha\right)$$

when $k \geq 4$ for all $\alpha \in [0,1]$. And then combined with the Hardy-Littlewood method, this enables us to give some short interval variants of Hua's theorems in additive prime number theory.

1. Introduction and statement of results

Let $\Lambda(n)$ be the von Mangoldt function, $k \geq 1$ an integer, x real and $2 \leq y \leq x$. The estimate of the exponential sum over primes in short intervals

$$S_k(x,y;\alpha) = \sum_{x<n\leq x+y} \Lambda(n) e\left(n^k\alpha\right) \tag{1.1}$$

was first studied by I. M. Vinogradov [11] in 1939 with his elementary method. Since then this topic has attracted the interest of quite a number of authors (see [1, 5, 6, 7, 8, 9, 10, 12] etc.). These sums arise naturally and play important roles when solving the Waring-Goldbach problems in short intervals by the circle method. In particular, the case $k = 1$, i.e., the linear exponential sum over primes in short intervals, was studied quite extensively, because of its applications to the study of the Goldbach-Vinogradov theorem with three almost equal prime variables (see [12] and the references therein).

For the case $k = 2$, Liu and Zhan [7] first established a non-trivial estimate of $S_2(x,y;\alpha)$ for all α and all published results before their result are valid only for α in a very thin subset of $[0,1]$. In [8], Lü and Lao improved the results in [7] to be as good as what was previously derived from the Generalized Riemann Hypothesis.

In this paper we deal with $S_k(x,y;\alpha)$ for all $\alpha \in [0,1]$ in the general case $k \geq 3$. In [6], Liu and Zhan first established a non-trivial estimate of $S_k(x,y;\alpha)$ for all $\alpha \in \mathbb{R}$ and $k \geq 3$. To state Liu and Zhan's result, we introduce some notation. Let $A > 0$ be any given large constant, $\varepsilon > 0$ sufficiently small. We further put

$$L = \log x, \quad P = L^{c_1}, \quad \mathcal{P} = x^{k\varrho}, \quad Q = \frac{y^{2k-1}}{x^{k-1}} L^{-c_3}, \quad R = yx^{k-1}L^{-c_2}, \tag{1.2}$$

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such that

$$2 \le 2P < 2\mathcal{P} < Q \le R \le x^k, \tag{1.3}$$

where ϱ is a positive parameter depending on k which will be specified later and c_i denote positive constants that depend at most on the positive numbers A, k and ε. By Dirichlet's lemma on rational approximation, any $\alpha \in [0,1]$ can be written as

$$\alpha = \frac{a}{q} + \lambda, \quad \text{with} \quad (a,q) = 1, \quad 1 \le a \le q \le Q, \quad |\lambda| \le \frac{1}{qQ}. \tag{1.4}$$

Then every $\alpha \in [0,1]$ given in the form of (1.4) satisfies one of the following three conditions:

$$(a) \qquad q \le P, \ |\lambda| \le \frac{1}{R};$$

$$(b) \qquad P < q \le Q, \ |\lambda| \le \frac{1}{qQ};$$

$$(c) \qquad q \le P, \ \frac{1}{R} < |\lambda| \le \frac{1}{qQ}.$$

Denote by $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ the three subsets of α satisfying (a), (b) and (c) respectively. Then $[0,1]$ is the disjoint union of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. The main result in [6] is the following

Theorem 0 (Liu-Zhan)**.** *Let $k \ge 3$, and $K = 2^{k-1}$. Then for any $A > 0$, there exist $c_1, c_2 > 0$, such that the estimate*

$$S_k(x,y;\alpha) = \begin{cases} M_k(x,y;\alpha) + O(yL^{-A}), & if\ q \le P,\ |\lambda| \le \frac{1}{R}; \\ O(yL^{-A}), & otherwise, \end{cases}$$

holds for

$$x^{\varphi+\varepsilon} \le y \le x,$$

where

$$\varphi = \varphi_k = \begin{cases} 1 - \frac{1}{K+1}, & if\ 3 \le k \le 5; \\ 1 - \frac{2}{k^2+3k+4}, & if\ k \ge 6, \end{cases}$$

and $M_k(x,y;\alpha)$ is the main term, which can be expressed as

$$M_k(x,y;\alpha) = \frac{1}{\varphi(q)} \sum_{\substack{h=1 \\ (h,q)=1}}^{q} e\left(\frac{ah^k}{q}\right) \int_x^{x+y} e(\lambda u^k) du.$$

Another result for the case $k \ge 3$ is given by Kumchev in [4]. His result is much better when q is large, but is not non-trivial for all $\alpha \in [0,1]$ (See [4, Theorem 1]). Hence in this paper we will combine the method used by Liu and Zhan with the method used by Kumchev to improve Theorem 0. Our main results of this paper are the following two theorems.

Theorem 1. *Let $k \ge 3$. Then for any $A > 0$, there exist $c_1, c_2 > 0$, such that the estimate*

$$S_k(x,y;\alpha) = \begin{cases} M_k(x,y;\alpha) + O(yL^{-A}), & if\ q \le P,\ |\lambda| \le \frac{1}{R}; \\ O(yL^{-A}), & otherwise, \end{cases}$$

holds for

$$x^{\vartheta+\varepsilon} \le y \le x,$$

where

$$\vartheta = \vartheta_k = \begin{cases} \frac{4}{5}, & if\ k = 3; \\ 1 - \frac{1}{2k}, & if\ k \ge 4, \end{cases}$$

The estimate given in Theorem 1, when combined with the Hardy-Littlewood method as in [6], enables us to give some short interval variants of Hua's theorems in additive prime number theory [3].

Theorem 2. *Let $k \ge 3$, $K = 2^{k-1}$, and*

$$\varsigma = \varsigma_k = \min\left\{1 - \frac{1}{k(K+1)}, 1 - \frac{1}{2k^2}\right\}. \tag{1.5}$$

Denote by $R_3(N, U)$ the number of solutions of the equation

$$\begin{cases} N = p_1 + p_2 + p_3^k, \\ |p_1 - \frac{N}{3}| \le U,\ |p_2 - \frac{N}{3}| \le U,\ |p_3^k - \frac{N}{3}| \le U. \end{cases}$$

Then for $U = N^{\varsigma+\varepsilon}$, we have

$$R_3(N, U) = 3^{2-1/k} C_3(N) \frac{U^2}{N^{1-1/k} \log^3 N} \left(1 + O\left(\frac{1}{\log N}\right)\right),$$

where

$$C_3(N) = \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^3(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{aN}{q}\right) \sum_{\substack{h=1 \\ (h,q)=1}}^{q} e\left(\frac{ah^k}{q}\right),$$

and $C_3(N) > c > 0$ for odd N.

Similarly, an almost-all result on the sum of a prime and a k-th power of a prime in short intervals can also be obtained.

Remark 1. The method in proving Theorem 1 can also be applied to establish a short interval estimate for exponential sums involving the Möbius function $\mu(n)$.

Notation. Throughout the paper, the letter ε denotes a sufficiently small positive real number, while c without subscript stands for an absolute positive constant; both of them may be different at each occurrence. For example, we may write

$$L^c L^c \ll L^c, \quad x^\varepsilon \ll y^\varepsilon.$$

Any statement in which ε occurs holds for each positive ε, and any implied constant in such a statement is allowed to depend on ε. The letter p, with or without subscripts, is reserved for prime numbers. We write $(a, b) = \gcd(a, b)$, and we use $m \sim M$ as an abbreviation for the condition $M < m \le 2M$.

2. Reduction of Theorem 1

For $\alpha \in \mathcal{A}$, the major arcs in the circle method, we need to show that

$$S_k(x,y;\alpha) = M_k(x,y;\alpha) + O(yL^{-A}). \tag{2.1}$$

Just as the treatment in [6], this can be easily established by the partial integration and the Siegel-Walfisz theorem in short intervals

$$\sum_{x<n\le x+y} \Lambda(n)\chi(n) = \delta_\chi y + O(yL^{-A}) \tag{2.2}$$

for $x^{\frac{7}{12}+\varepsilon} \le y \le x$ and any character χ modulo $q \le L^C$, where $\delta_\chi = 1$ if χ is principal and $\delta_\chi = 0$ otherwise, $C > 0$ is any constant.

Hence for the proof of Theorem 1 reduces to show that

$$S_k(x,y;\alpha) \ll yL^{-A}, \quad \alpha \in \mathcal{B} \cup \mathcal{C}, \tag{2.3}$$

with suitable choice of constants c_i $(i = 1,2,3)$ in (1.2).

For $\alpha \in \mathcal{B}$, in order to improve the result, we further divide the set $\mathcal{B}$ into two subsets

$$\mathcal{B}_1 = \left\{ \alpha \in [0,1] \,\middle|\, \alpha = \frac{a}{q} + \lambda,\ (a,q) = 1,\ P < q < \mathcal{P},\ |\lambda| \le \frac{1}{qQ} \right\} \tag{2.4}$$

and

$$\mathcal{B}_2 = \left\{ \alpha \in [0,1] \,\middle|\, \alpha = \frac{a}{q} + \lambda,\ (a,q) = 1,\ \mathcal{P} \le q \le Q,\ |\lambda| \le \frac{1}{qQ} \right\} \tag{2.5}$$

where $\varrho = \varrho_k$ is a small parameter satisfying some conditions which will be given later. Then we estimate $\alpha \in \mathcal{B}_1$ and $\alpha \in \mathcal{B}_2$ separately.

For $\alpha \in \mathcal{B}_1$, we establish the following proposition

Proposition 1. *Let $k \ge 3$ and $\varrho < \frac{k-1}{2k^2}$. Then there exist $c_1, c_3 > 0$, such that*

$$S_k(x,y;\alpha) \ll yL^{-A}$$

holds for $\alpha \in \mathcal{B}_1$ and

$$x^{\beta+\varepsilon} \le y \le x, \quad \textit{with} \ \ \beta = \beta_k = 1 - \frac{1}{2k}. \tag{2.6}$$

For $\alpha \in \mathcal{B}_2$, our result is

Proposition 2. *Let $k \ge 4$, $\varrho < \frac{1}{k^3}$ and c_1, c_3 be fixed according to the discussion above. Then the estimate*

$$S_k(x,y;\alpha) \ll yL^{-A}$$

holds for $\alpha \in \mathcal{B}_2$ and

$$x^{\gamma+\varepsilon} \le y \le x^{\omega}, \quad \textit{with} \ \ \gamma = \gamma_k = 1 - \frac{1}{2k-1}, \ \ \omega = \omega_k = 1 - \frac{1}{k^3}. \tag{2.7}$$

Remark 2. Actually the precise choice of ω is unimportant and we just need $\varphi_k < \omega_k < 1$. The estimate can be improved to be the form of $y^{1-\rho+\varepsilon}$ if we use the method in [4] to give the estimate for exponential sums of Type II instead of Lemma 6 below. But since it has no influence on our main result, we will not do it.

In proving the above propositions, we estimate the exponential sums of type I and type II respectively. At first we will give the estimate of the exponential sums of type II

$$\sum_{m\sim M} \sum_{\substack{n\sim N \\ x<mn\le x+y}} a(m)b(n)e\left((mn)^k\alpha\right) \tag{2.8}$$

for all $\alpha \in \mathcal{B}$ which is Proposition A in [6], where $a(m)$, $b(n)$ are any real numbers satisfying that $a(m) \ll \tau_\ell(m)L$, $b(n) \ll \tau_\ell(n)L$ and $\tau_\ell(n)$ is the number of ordered factorizations of n as the product of exactly ℓ positive integers. Then for the exponential sums of type I

$$\sum_{m\sim M} a(m) \sum_{\substack{n\sim N \\ x<mn\le x+y}} e\left((mn)^k\alpha\right), \tag{2.9}$$

when $\alpha \in \mathcal{B}_1$, we apply van der Corput's method to handle them and a theorem of Hua to concern complete exponential sums as [6] does; when $\alpha \in \mathcal{B}_2$, we employ the method used by Kumchev in [4] to deal with them. At last we can deduce the results by appealing to the Heath-Brown's identity.

For $\alpha \in \mathcal{C}$, we have the following result

Proposition 3. *Let $k \ge 3$, and c_1, c_3 be fixed according to the discussion above. Then there exists $c_2 > 0$ such that the estimate*

$$S_k(x, y; \alpha) \ll yL^{-A}$$

holds for $\alpha \in \mathcal{C}$ and

$$x^{\eta+\varepsilon} \le y \le x, \quad \eta = \eta_k = 1 - \frac{1}{2k-1}. \tag{2.10}$$

Proof. See [6, Theorem 6]. □

We conclude from Propositions 1, 2 and 3 that

Corollary 1. *The estimate (2.3) holds subject to the condition $x^{\beta+\varepsilon} \le y \le x^{\omega}$.*

It is easily seen that Theorem 1 follows form Theorem 0 and Corollary 1.

In Section 3 we shall give some lemmas which will be used later. In Section 4 and 5 we shall give the proof of Proposition 1 and 2 respectively.

3. Lemmas

Lemma 1 (van der Corput)**.** *Let $f(x)$ be a real differentiable function on $[a,b]$, $f'(x)$ be monotonic and $|f'(x)|\le\delta$, $0<\delta<1$. Then*

$$\sum_{a<n\le b} e(f(n)) = \int_a^b e(f(x))dx + O\left(\frac{1}{1-\delta}\right).$$

Proof. See [6, Lemma 6.4]. □

Lemma 2 (Hua)**.** *Let $f(x)=a_1x+\cdots+a_kx^k$ be a polynomial with integer coefficients, and $(a_1,\ldots,a_k,q)=d$. Then*

$$\sum_{1\le n\le q} e\left(\frac{f(n)}{q}\right) \ll_{k,\varepsilon} q^{1-\frac{1}{k}+\varepsilon}d^{\frac{1}{k}}.$$

Proof. See [3, Theorem 2]. □

Lemma 3 (Heath-Brown)**.** *Let $z\ge1$ and $J\ge1$. Then for any $n\le 2z^J$, we have*

$$\Lambda(n)=\sum_{j=1}^{J}(-1)^{j-1}\binom{J}{j}\sum_{\substack{n_1n_2\cdots n_{2j}=n\\ n_{j+1},\ldots,n_{2j}\le z}}\cdots\sum(\log n_1)\mu(n_{j+1})\cdots\mu(n_{2j}). \tag{3.1}$$

Proof. See [2, Section 2]. □

Lemma 4. *Let $k\ge3$, we define the multiplicative function $w_k(q)$ by*

$$w_k\left(p^{ku+v}\right)=\begin{cases} kp^{-u-1/2}, & \text{if } u\ge0,\ v=1,\\ p^{-u-1}, & \text{if } u\ge0,\ v=2,\ldots,k.\end{cases}$$

Then we have

$$\sum_{n\sim N} w_k\left(\frac{q}{(q,n^j)}\right) \ll q^{\varepsilon}w_k(q)N \qquad (1\le j\le k). \tag{3.2}$$

Proof. See [4, Lemma 2.1]. □

Lemma 5 (Kumchev)**.** *Let $k\ge3$ ba an integer and let $0<\rho\le\sigma_k$, where $\sigma_k=\max\left\{\frac{1}{K},\frac{1}{2k(k-2)}\right\}$. Suppose that $y\le x$, $x^k\le y^{k+1-2\rho}$. Then either*

$$\sum_{x<n\le x+y} e\left(n^k\alpha\right) \ll y^{1-\rho+\varepsilon}, \tag{3.3}$$

or there exist integers a and q such that

$$1\le q\le y^{k\rho},\quad (a,q)=1,\quad |q\alpha-a|\le x^{1-k}y^{k\rho-1}, \tag{3.4}$$

and

$$\sum_{x<n\le x+y} e\left(n^k\alpha\right) \ll \frac{w_k(q)y}{1+yx^{k-1}|\alpha-a/q|}+x^{k/2+\varepsilon}y^{(1-k)/2}. \tag{3.5}$$

Proof. See [4, Lemma 2.2]. □

4. THE CASE $\alpha \in \mathcal{B}_1$

The aim of this section is to give the proof of Proposition 1. At first, we give the following estimate of the exponential sums of type II which will be used for all the cases when $\alpha \in \mathcal{B}$.

Lemma 6. *Let $M, N \geq 1$, $x \ll MN \ll x$ and define*

$$\mathcal{T}_2 = \sum_{m \sim M} \sum_{\substack{n \sim N \\ x < mn \leq x+y}} a(m)b(n)e\left((mn)^k \alpha\right).$$

Then we have

$$\mathcal{T}_2 \ll yL^{-A} \tag{4.1}$$

holds for

$$xy^{-1}L^{c_4} \leq M \leq yL^{-c_4}, \quad L^{c_5} \leq q \leq y^{2k-1}x^{1-k}L^{-c_5}, \tag{4.2}$$

where $c_j = c_j(A) > 0,\ j = 4, 5$.

Proof. See [6, Propostion A]. □

For $\alpha \in \mathcal{B}_1$, we need the following lemma to treat the exponential sums of type I. The proof is similar to [6, Propostion C].

Lemma 7. *Let $M, N \geq 1$, $x \ll MN \ll x$ and define*

$$\mathcal{T}_1 = \sum_{m \sim M} a(m) \sum_{\substack{n \sim N \\ x < mn \leq x+y}} e\left((mn)^k \alpha\right).$$

Then we have

$$\mathcal{T}_1 \ll yL^{-A} \tag{4.3}$$

holds for

$$M \ll \min\left\{\frac{Q}{x^{k-1}L}, \frac{y}{\mathcal{P}}L^{-c_6}\right\}, \quad L^{c_1} < q \leq \mathcal{P}, \tag{4.4}$$

with c_1, c_6 sufficiently large.

Proof. We begin with the estimation of the inner sum

$$S_m = \sum_{X < n \leq X+Y} e\left(m^k n^k \alpha\right), \tag{4.5}$$

where $X,\ Y$ satisfy that

$$X = \max\left\{\frac{x}{m}, N\right\} \asymp \frac{x}{M},$$

$$Y = \min\left\{\frac{x+y}{m}, 2N\right\} - \max\left\{\frac{x}{m}, N\right\} \ll \frac{y}{M}$$

with $m \sim M$. It is easy to see that

$$\begin{aligned} S_m &= \sum_{v\le q} \sum_{X<qu+v\le X+Y} e\left(m^k(qu+v)^k\alpha\right) \\ &= \sum_{v\le q} e\left(\frac{am^kv^k}{q}\right) \sum_{X<qu+v\le X+Y} e\left(m^k(qu+v)^k\lambda\right). \end{aligned} \tag{4.6}$$

Since $M \ll \frac{Q}{x^{k-1}L}$, we have

$$\frac{d}{du}(m^k(qu+v)^k\lambda) = km^k(qu+v)^{k-1}q\lambda \ll \frac{M^kX^{k-1}}{Q} \ll \frac{Mx^{k-1}}{Q} < \frac{1}{2}.$$

We can thus apply Lemma 1, which ensures that the inner sum on the right hand side of (4.6) is

$$\begin{aligned} &= \int_{\frac{X-v}{q}}^{\frac{X+Y-v}{q}} e\left(m^k(qu+v)^k\lambda\right) du + O(1) \\ &= \frac{Y}{q}\int_{\frac{X}{Y}}^{\frac{X+Y}{Y}} e\left(m^k(Yu)^k\lambda\right) du + O(1). \end{aligned}$$

Hence (4.6) becomes

$$S_m = \frac{Y}{q}\int_{\frac{X}{Y}}^{\frac{X+Y}{Y}} e\left(m^k(Yu)^k\lambda\right) du \sum_{v\le q} e\left(\frac{am^kv^k}{q}\right) + O(q).$$

From this and Lemma 2, we conclude that

$$\begin{aligned} \mathcal{T}_1 &\ll \sum_{m\sim M} |a(m)||S_m| \\ &\ll \frac{y}{Mq}\sum_{m\sim M} \tau_\ell(m)L\left|\sum_{v\le q} e\left(\frac{am^kv^k}{q}\right)\right| + qML^c \\ &\ll yM^{-1}q^{-\frac{1}{k}+\varepsilon}\sum_{m\sim M}\tau_\ell(m)(m^k,q)^{\frac{1}{k}} + qML^c. \end{aligned}$$

Since

$$\begin{aligned} \sum_{m\sim M}\tau_\ell(m)(m^k,q)^{\frac{1}{k}} &\le \sum_{m\sim M}\tau_\ell(m)(m,q) \le \sum_{\substack{d|q\\ d\le 2M}}\sum_{\substack{m\sim M\\ d|m}}\tau_\ell(m)d \\ &\ll \sum_{d|q} d\tau_\ell(d)\frac{M}{d}L^c \ll ML^c\tau_\ell(q)\tau(q), \end{aligned}$$

we have

$$\mathcal{T}_1 \ll yq^{-\frac{1}{k}+\varepsilon} + qML^c,$$

which gives the desired result on taking c_1, c_6 sufficiently large. □

Remark 3. Let

$$\mathcal{T}_1^* = \sum_{m\sim M} a(m) \sum_{\substack{n\sim N \\ x<mn\le x+y}} e\left((mn)^k\alpha\right)\log n. \tag{4.7}$$

Under the condition of Lemma 7 we have

$$\mathcal{T}_1^* \ll yL^{-A}.$$

Utilizing Lemma 6 and Lemma 7, we now establish Proposition 1 via Heath-Brown's identity.

Proof of Proposition 1. Applying the Heath-Brown identity we obtain that the exponential sum $S_k(x,y;\alpha)$ can be written as $O(L^c)$ linear combinations of

$$\Sigma = \sum_{n_1\sim N_1}\cdots\sum_{\substack{n_{2J}\sim N_{2J} \\ x<n_1\cdots n_{2J}\le x+y}} a_1(n_1)\cdots a_{2J}(n_{2J})e\left((n_1\cdots n_{2J})^k\alpha\right)$$

where

$$a_1(n_1)=\log n;\quad a_j(n)=1,\ 2\le j\le J;\quad a_j(n)=\mu(n),\ J+1\le j\le 2J$$

and

$$x\ll N_1\cdots N_{2J}\ll x;\quad N_j\ge\frac{1}{2},\ 1\le j\le 2J;\quad N_j\ll 2x^{\frac{1}{J}},\ J+1\le j\le 2J.$$

To prove Proposition 1, we take $J=2k$ and $\varrho<\frac{k-1}{2k^2}$. Then we have

$$x^{\frac{1}{J}}<\min\left\{\frac{Q}{x^{k-1}L},\ \frac{y}{\mathcal{P}}L^{-c_6}\right\}.$$

The analysis involves several cases depending on the sizes of $N_1,\dots,N_{2J}$.

Case 1: If there exists $1\le j\le 2J$ such that $N_j>x^{\frac{J-1}{J}}$, then it follows that $1\le j\le J$. In this case Σ can be written in the form of $\mathcal{T}_1$ in Lemma 7 or $\mathcal{T}_1^*$ in (4.7) with $M=\prod_{i\ne j}N_i\le x^{\frac{1}{J}}$ satisfying (4.4). Hence Proposition 1 is true.

Case 2: If there exists $1\le j\le 2J$ satisfying $2x^{\frac{1}{J}}<N_j\le x^{\frac{J-1}{J}}$, we also have $1\le j\le J$. In this case Σ can be written as $\mathcal{T}_2$ in Lemma 6 with $M=N_j$ satisfying (4.2). Proposition 1 then follows in this case.

Case 3: It remains to consider the case $N_j\le 2x^{\frac{1}{J}}$ for all $1\le j\le 2J$. Take the smallest i such that

$$N_1\cdots N_i>2x^{\frac{1}{J}}.$$

Since $N_1\le 2x^{\frac{1}{J}}$, we have $i\ge 2$ and

$$N_1\cdots N_i=(N_1\cdots N_{i-1})N_i<2x^{\frac{1}{J}}2x^{\frac{1}{J}}=4x^{\frac{2}{J}}.$$

Let $M=N_1\cdots N_i$. Then M satisfies the condition of Lemma 6. This completes the proof of Proposition 1.

□

5. The case $\alpha \in \mathcal{B}_2$

In this section, we will use the method in [4] to prove Proposition 2. At first, we will establish the following lemma which will be used to deal with the exponential sums of type I with $\alpha \in \mathcal{B}_2$.

Lemma 8. *Let $k \geq 3$, $0 < \rho < \sigma_k/2$ and $\rho < \frac{1}{k^3}$. Let $M, N \geq 1$, $x \ll MN \ll x$. Suppose that α is real that there exist integers a and q such that (1.4) holds with Q given by (1.2). Let $a(m) \leq \tau_\ell(m)L$, and define*

$$\mathcal{T}_1 = \sum_{m \sim M} a(m) \sum_{\substack{n \sim N \\ x < mn \leq x+y}} e\left((mn)^k \alpha\right).$$

Then

$$\mathcal{T}_1 \ll y^{1-\rho+\varepsilon} + \frac{w_k(q) y x^{\varepsilon}}{1 + y x^{k-1}|\alpha - a/q|},$$

provided that

$$M \ll \min\left\{ y^{\frac{(1-\rho)(k+1-2\rho)}{1-2\rho}} x^{\frac{-k}{1-2\rho}},\ y^{k+1-2\rho} x^{-k},\ x^{1-\frac{k+1}{k}\frac{\rho}{\sigma_k}} \right\}, \quad M^{2k} \ll x^{k-2k\rho-\frac{1-2\rho}{k+1-2\rho}}, \tag{5.1}$$

and

$$x^{\gamma+\varepsilon} \leq y \leq x^{\omega}, \tag{5.2}$$

with

$$\gamma = \gamma_k = 1 - \frac{1}{2k-1}, \quad \text{and} \quad \omega = \omega_k = 1 - \frac{1}{k^3}. \tag{5.3}$$

Proof. Set

$$S_m = \sum_{X < n \leq X+Y} e\left(m^k n^k \alpha\right),$$

where $X,\ Y$ satisfy that

$$X = \max\left\{\frac{x}{m}, N\right\} \asymp \frac{x}{M},$$
$$Y = \min\left\{\frac{x+y}{m}, 2N\right\} - \max\left\{\frac{x}{m}, N\right\} \ll \frac{y}{M}$$

with $m \sim M$. Denote $\mathcal{M}_0$ to be the set of m, with $m \sim M$, for which satisfy that

$$Y^{k+1-2\rho} \geq X^k.$$

So, by (5.1) and (5.2), we have

$$\begin{aligned} \mathcal{T}_1 &\ll ML^c X^{\frac{k}{k+1-2\rho}} + \sum_{m \in \mathcal{M}_0} a(m) \sum_{\substack{n \sim N \\ x < mn \leq x+y}} e\left((mn)^k \alpha\right) \\ &\ll y^{1-\rho+\varepsilon} + \sum_{m \in \mathcal{M}_0} a(m) S_m. \end{aligned}$$

Define ν by $Y^\nu = x^\rho L^{-1}$ for any fixed $m \in \mathcal{M}_0$. Note that, by (5.1), we have

$$\nu < \sigma_k.$$

We denote by $\mathcal{M}$ the set of integers $m \in \mathcal{M}_0$, for which there exist integers b_1 and r_1 with

$$1 \le r_1 \le Y^{k\nu}, \quad (b_1, r_1) = 1, \quad |r_1 m^k \alpha - b_1| \le X^{1-k} Y^{k\nu - 1}. \tag{5.4}$$

We apply Lemma 5 to the summation over n and get

$$S_m \ll Y^{1-\nu+\varepsilon} + \frac{w_k(r_1) Y}{1 + Y X^{k-1} |m^k \alpha - b_1/r_1|} + X^{k/2+\varepsilon} Y^{(1-k)/2},$$

for $m \in \mathcal{M}$. So

$$\begin{aligned} \mathcal{T}_1 &\ll y^{1-\rho+\varepsilon} + \sum_{m \sim M} Y^{1-\nu+\varepsilon} \\ &\quad + \sum_{m \in \mathcal{M}} a(m) \left(\frac{w_k(r_1) Y}{1 + Y X^{k-1} |m^k \alpha - b_1/r_1|} + X^{k/2+\varepsilon} Y^{(1-k)/2} \right). \end{aligned}$$

Then by (5.1) we have

$$\mathcal{T}_1 \ll y^{1-\rho+\varepsilon} + T_1(\alpha),$$

where

$$T_1(\alpha) = \sum_{m \in \mathcal{M}} \frac{a(m) w_k(r_1) Y}{1 + Y X^{k-1} |m^k \alpha - b_1/r_1|}.$$

We apply Dirichlet's theorem on Diophantine approximation to find integers b and r with

$$1 \le r \le x^{-k\rho} Y X^{k-1}, \quad (b, r) = 1, \quad |r\alpha - b| \le x^{k\rho} Y^{-1} X^{1-k}. \tag{5.5}$$

By (5.1), (5.4) and (5.5), we have

$$\begin{aligned} |b_1 r - b m^k r_1| &= |r(b_1 - r_1 m^k \alpha) + r_1 m^k (r\alpha - b)| \\ &\le x^{-k\rho} Y X^{k-1} X^{1-k} Y^{k\nu-1} + Y^{k\nu} (2M)^k x^{k\rho} Y^{-1} X^{1-k} \\ &\ll L^{-k} + M^{2k - \frac{1-2\rho}{k+1-2\rho}} L^{-k} x^{2k\rho - k + \frac{1-2\rho}{k+1-2\rho}} \\ &\ll L^{-k} < 1, \end{aligned}$$

whence

$$\frac{b_1}{r_1} = \frac{m^k b}{r}, \quad r_1 = \frac{r}{(r, m^k)}.$$

Thus, by Lemma 4, we have

$$\begin{aligned} T_1(\alpha) &\le \sum_{m \in \mathcal{M}} \frac{a(m) w_k\left(\frac{r}{(r,m^k)}\right) Y}{1 + Y X^{k-1} m^k |\alpha - b/r|} \\ &\ll \frac{y M^{-1+\varepsilon}}{1 + y x^{k-1} |\alpha - b/r|} \sum_{m \sim M} w_k \left(\frac{r}{(r, m^k)} \right) \\ &\ll \frac{y M^{-1+\varepsilon}}{1 + y x^{k-1} |\alpha - b/r|} r^{\varepsilon} w_k(r) M \\ &\ll \frac{w_k(r) y x^{\varepsilon}}{1 + y x^{k-1} |\alpha - b/r|}. \end{aligned}$$

Recall that b and r satisfy the conditions (5.5). We now consider three cases depending on the size of r and $|r\alpha - b|$.

Case 1: If $r > x^{k\rho}L^{-1}$, then $w_k(r) \ll (x^{k\rho}L^{-1})^{-1/k}$. Hence $T_1(\alpha) \ll y^{1-\rho+\varepsilon}$.
Case 2: If $r \le x^{k\rho}L^{-1}$ and $|r\alpha - b| > y^{-1}x^{1-k}x^{(k+1)\rho}L^{-1}$, then $T_1(\alpha) \ll y^{1-\rho+\varepsilon}$.
Case 3: If $r \le x^{k\rho}L^{-1}$ and $|r\alpha - b| \le y^{-1}x^{1-k}x^{(k+1)\rho}L^{-1}$. We have

$$\begin{aligned}|ra - bq| &= |r(a - q\alpha) + q(r\alpha - b)| \\ &\le x^{k\rho}L^{-1}\frac{1}{Q} + Qy^{-1}x^{1-k}x^{(k+1)\rho}L^{-1} \\ &\le \frac{x^{k-1+k\rho}}{y^{2k-1}} + \frac{y^{2k-2}}{x^{2k-2}}x^{(k+1)\rho}L^{-1}.\end{aligned}$$

Since $\rho < \frac{1}{k^3}$, by (5.2), we have $|ra - bq| < 1$, hence

$$a = b, \quad q = r.$$

Then

$$T_1(\alpha) \ll \frac{w_k(q)yx^{\varepsilon}}{1 + yx^{k-1}|\alpha - a/q|}.$$

So we prove

$$\mathcal{T}_1 \ll y^{1-\rho+\varepsilon} + \frac{w_k(q)yx^{\varepsilon}}{1 + yx^{k-1}|\alpha - a/q|}.$$

□

Remark 4. Let

$$\mathcal{T}_1^* = \sum_{m\sim M} a(m) \sum_{\substack{n\sim N \\ x<mn\le x+y}} e\left((mn)^k\alpha\right)\log n. \tag{5.6}$$

Under the condition of Lemma 8 we have

$$\mathcal{T}_1^* \ll y^{1-\rho+\varepsilon} + \frac{w_k(q)yx^{\varepsilon}}{1 + yx^{k-1}|\alpha - a/q|}.$$

Remark 5. One can estimate the exponential sums

$$\sum_{m_1\sim M_1}\sum_{m_2\sim M_2} a(m_1, m_2) \sum_{\substack{n\sim N \\ x<m_1m_2n\le x+y}} e\left((m_1m_2n)^k\alpha\right)$$

with some suitable conditions on M_1 and M_2 as [4, Lemma 3.2] did, and then can give a better result then Lemma 8. Since it has no influence on our main results, we will not do it.

Utilizing Lemma 6 and Lemma 8, we can establish Proposition 2 via Heath-Brown's identity.

Proof of Proposition 2. For $k \geq 4$, take $J = 2k-1$. Since $\sigma_k = \max\left\{\frac{1}{K}, \frac{1}{2k(k-2)}\right\}$, $0 < \varrho = \rho < \frac{1}{k^3}$ and $y \geq x^{1-\frac{1}{2k-1}+\varepsilon}$, we have

$$x^{\frac{1}{J}} \ll \min\left\{y^{\frac{(1-\rho)(k+1-2\rho)}{1-2\rho}} x^{\frac{-k}{1-2\rho}},\ y^{k+1-2\rho}x^{-k},\ x^{1-\frac{k+1}{k}\frac{\rho}{\sigma_k}},\ x^{\frac{1}{2k}\left(k-2k\rho-\frac{1-2\rho}{k+1-2\rho}\right)}\right\}.$$

To estimate $S_k(x, y; \alpha)$, we now apply Lemma 3 with $z = x^{\frac{1}{J}}$. Then we get the desire result by the same argument as the proof of Proposition 1. $\square$

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